Homework 1: fundamentals

Exercise 1: Calculus

Check the assertion from class that

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is a smooth function. A more specific and helpful claim is that there is a sequence of polynomials $P_k(t)$ so that

$$f^{(k)}(x) = \begin{cases} P_k(\frac{1}{x})e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Exercise 2: de Rham cohomology with compact support

Let ω be a k-form on $\mathbb{R}^n$. We say that ω is compactly supported if there is some compact set K so that $\omega = 0$ on $\mathbb{R}^n \setminus K$. Write $\Omega_c^*(\mathbb{R}^n)$ for the set of compactly supported k-forms.

- Show that $\Omega_c^*(\mathbb{R}^n)$ is a subalgebra of $\Omega^*(\mathbb{R}^n)$.
- Show that the exterior derivative operator d restricts to an operation

$$d : \Omega_c^*(\mathbb{R}^n) \to \Omega_c^*(\mathbb{R}^n).$$

- Define the compactly supported de Rham cohomology of $\mathbb{R}^n$ to be

$$H^k_{c,dR}(\mathbb{R}^n) = \frac{\ker(d|_{\Omega_c^k})}{(d|_{\Omega_c^{k-1}})}.$$

  Compute $H^k_{c,dR}(\mathbb{R}^0)$ and $H^k_{c,dR}(\mathbb{R}^1)$. Compare your results and methods with the ones we used in class.

Exercise 3: Smooth function hygiene

- Let M be a smooth manifold, let $U \subset M$, and let $\phi : U \to \mathbb{R}^k$ be a chart. Let $f : U \to \mathbb{R}^m$ be a continuous function. Suppose that $f \circ \phi^{-1}$ is smooth.
  Let $\psi : U \to \mathbb{R}^k$ be another chart with the same domain. Show that $f \circ \phi^{-1}$ is smooth.
- A function $f : M \to \mathbb{R}^k$ is <u>smooth</u> if for every $p \in M$ there is a smooth chart (U, ϕ) with $p \in U$ so that $f \circ \phi^{-1}$ is smooth on $\phi(U)$. Show that if f is smooth, then $f \circ \psi^{-1}$ is smooth for <u>any</u> chart (V, ψ) around p.
- Show that the composition of smooth functions between manifolds is smooth. Show that the composition of diffeomorphisms is again a diffeomorphism.
- Write down a smooth function $\mathbb{R} \to \mathbb{R}$ which is invertible but not a diffeomorphism.

Exercise 4: Stereographic projection

In this exercise you will show that S^n is a manifold which can be covered by two charts. Think of $S^n \subset \mathbb{R}^{n+1}$ as the set

$$S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1 : (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}\}$$

Write $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$. (These are the <u>north</u> and <u>south poles</u>.)

- Define a map $p : S^n \setminus N \to \mathbb{R}^n$ as follows. Let $x \in S^n \setminus N$. There is a unique line ℓ_x between N and X. This line intersects the plane $\{x_{n+1} = 0\}$ in one point. This point is $p(x)$. Identify $\mathbb{R}^n$ with the set

$$\{(x_1, \ldots, x_{n+1}) : x_{n+1} = 0\}$$

  so that p really is a map to $\mathbb{R}^n$.
  First, check that $p(x)$ is well-defined: why does ℓ_x intersect $\{x_{n+1} = 0\}$ in exactly one point?
- Write down a nice expression for $p(x)$ in terms of $(x_1, \ldots, x_{n+1})$. (Hint: your expression shouldn't make sense if you try to set $x = N$.)
- Show that p has an inverse and write down a nice expression for it.
- $p(\mathbb{R}^n)$ covers most of the sphere. Find another good chart which covers the rest.
- Show that S^n cannot be covered by a single chart.

Exercise 5

Prove (not just by picture!) that the union of the x- and y-axes in $\mathbb{R}^2$ is not a manifold.

Exercise 6: Products and graphs

Let X and Y be smooth manifolds. Show that $X \times Y$ is a smooth manifold. Suppose that f and g are smooth functions with domains X and Y, respectively. Show that $(f \times g)(x, y) = (f(x), g(y))$ is a smooth function.
Now suppose that $h : X \to Y$. Show that the graph of h,

$$\text{graph}(h) = \{(x, h(x)) : x \in X\}$$

is a smooth manifold. (Hint: define a map $X \to X \times Y$ which restricts to a diffeomorphism on the graph.)