Homework 2

Exercise 1

Let $\phi, \psi : U \to \mathbb{R}^n$ be two coordinate charts on a smooth manifold M. Let $m \in U$. Let $v \in T_m U$ be given in ϕ-coordinates by $\sum v_i \frac{\partial}{\partial x_i}$ and in ψ-coordinates by $\sum w_i \frac{\partial}{\partial y_i}$. Determine w_i in terms of v_i and the maps $\phi \circ \psi^{-1}$, $\psi \circ \phi^{-1}$. (Start by making sure you understand what's meant by ϕ-coordinates.)

Exercise 2

Let M, ϕ, ψ, m, and U be as above. Let ω be a k-form on U so that $\omega = \sum_I f_I dx_I$ in ϕ-coordinates and $\omega = \sum_J g_J dy_J$ in ψ-coordinates. Determine g_J in terms of f_I, ϕ, and ψ.

Exercise 3

On the last homework, you showed that $M \times N$ gets a smooth structure from M and N. There are smooth maps $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ given by projecting onto the first or second coordiante. (You do not have to check that they are smooth.)
Use these maps to cook up an isomorphism $T(p, q)(M \times N) \cong T_p M \times T_q N$.

Exercise 4

Let $F : M \to N$ be a smooth map of smooth manifolds so that $F_* : T_p \to T_{F(p)}$ is the zero map for all p. Show that F must be constant if M is connected.

Exercise 5

- Prove that the exterior derivative is not dependent on the choice of charts by proving the following coordinate-free formula. Let M be a smooth manifold. Let $\omega \in \Omega^p(M)$, and let $X^0, \dots, X^p$ be vector fields on M. Then

$$(d\omega)(X^0, \dots, X^p) = \sum_{i=0}^{p} (-1)^i X_i \omega(X^0, \dots, \widehat{X^i}, \dots, X^p) + \sum_{i<j} (-1)^{(i+1)} \omega([X^i, X^j], X^0, \dots, \widehat{X^i}, \dots, \widehat{X^j}, \dots, X^p)$$

  where summing over something with a hat means skipping it. (To clarify: $\omega(X^0, \dots, \widehat{X^i}, \dots, X^p)$ is a function from $M \to \mathbb{R}$ – why?)
  Prove this formula. (Hint: you need to compare this formula to the one in coordinates. So you'd really like to be able to replace the X^i with constant vector fields (defined only near m) like $\partial/\partial x_i$. Start by showing that if $X^0(m) = Y^0(m)$, then $(d\omega)(Y^0, X^1, \dots, X^p) = (d\omega)(X^0, \dots, X^p)$.)
- Your classmate Adam is confused by the previous problem: "why are we thinking about vector fields in the first place? The hint tells us that we only need to think about the value of X^0 at m. So we may as well just define $(d\omega)$ on a $(p+1)$-tuple of vectors. Our professor gives us obsfuscated exercises!" Explain why Adam is wrong, at least about the mathematics.

Exercise 6

Show that $H_{0,dR}(M)$ is determined by the point-set topology of M rather than the smooth structure on M. (To do so, describe $H_{0,dR}(M)$ in terms of the point-set topology of M.)

Exercise 7

Compute $H_{k,dR}(S^n)$ for $k \in \mathbb{N}$.