Homework 3

1. (Page 37 of B&T) Let $S^n(r)$ be the sphere of radius r in $\mathbb{R}^{n+1}$. Let

$$\omega = \frac{1}{r}\sum_{i=1}^{n+1}(-1)^{i-1}x_i dx_1 \cdots \widehat{dx_i} \cdots dx_{n+1}$$

be an n-form. (The hat denotes omission, i.e. $dx_1\widehat{dx_2}dx_3 = dx_1 dx_3$.)

   (a) Compute $\int_{S^n(1)} \omega$. (There might be a clever way to do this but at some point you have to really compute an integral.) Use your computation to conclude that ω is not an exact form.

   (b) Think of r the radius function on $\mathbb{R}^{n+1} \setminus \{0\}$. Show that $dr \wedge \omega = dx_1 \dots dx_n$.

   (c) B&T claim that your answer to the first part gives you a generator of $H^2(S^2)$. Explain why and give a formula for the generator of $H^n(S^n)$.

2. (Page 38 of B&T) Show that the map π_* in the proof of the Poincarèlemma is a chain map.

3. Compute the cohomology groups (regular and compactly supported) of the cylinder $S^1 \times \mathbb{R}$ and the open Mobius strip, i.e. the twisted $S^1 \times \mathbb{R}$. Do not use a Künneth formula.

4. Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a proper map. In class showed that $\deg(f) = \sum_{p\in f^{-1}(q)} \pm 1$ where the signs depend on p.

   (a) Show that if $f \simeq g$ then $\deg(f) = \deg(g)$ using differential forms.

   (b) The degree of f mod two is $\deg(f) \mod 2$. Give another proof that the degree mod two is homotopy invariant as follows. Let F be a homotopy between f and g. Consider the restriction of F to $\mathbb{R}^m \times [0,1]$. Study $F^{-1}(q)$. (To get started: what sort of space is $F^{-1}(q) \cap \mathbb{R}^m \times \{t\}$ for $t \in [0,1]$?)

5. Degree examples:

   (a) The unit circle in $\mathbb{C}$ is the set $\|z\| = 1$. The map $z \mapsto z^k$ for $k \in \mathbb{N}$ induces a smooth map $S^1 \to S^1$. Compute the degree of this map.

   (b) A complex polynomial $p(z)$ induces a smooth map $\mathbb{C} \to \mathbb{C}$. Show that the degree of this map is the degree of the polynomial.

   (c) (The linking number) Let M and N be disjoint manifolds in $\mathbb{R}^{r+1}$. (We haven't talked much about embedded manifolds. You can take M to be a subspace of $\mathbb{R}^{r+1}$ so that each $p \in M$ has a neighborhood U so that $U \cap M$ is diffeomorphic to $\mathbb{R}^m$.) Define

$$\lambda\colon M \times N \to \mathbb{R}^{r+1}$$

   by

$$\lambda(x,y) = \frac{x-y}{\|x-y\|}.$$

Verify that $\lambda(x, y)$ lands in the unit sphere in $\mathbb{R}^{r+1}$ for any x and y. Show that $\lambda(x, y) = -\lambda(y, x)$.

6. Let M be a connected manifold of dimension n. A <u>smooth curve in M</u> is a smooth map $[0, 1] \to M$. A <u>smooth loop in M</u> is a smooth map $\gamma\colon [0, 1] \to M$ so that $\gamma(0) = \gamma(1)$. (Of course this is equivalent to a smooth map $S^1 \to M$.)

   Show that if M is simply-connected then $H^1(M) = 0$.