Homework 3

1. (Exercise 6.2, page 54 from B&T) Show that two vector bundles on a manifold M are isomorphic if and only if their cocycles are equivalent (relative to some open cover).

2. (Exercise 6.43, page 65 from B&T) Let $\pi\colon E \to M$ be a rank two bundle. Then there is an isomorphism $H^*(M) \to H^{*+2}_{cv}(E)$ given by the wedge product with the Thom class Φ. It follows that any class $\omega \in H^{*+2}_{cv}(E)$ can be written $\Phi \wedge \pi^* u$ where $u \in H^*(M)$.

   Give a nice description of the class u so that $\Phi \wedge \pi^* u = \Phi \wedge \Phi$.

3. (Exercise 6.45, page 77 from B&T) There is a special bundle on $\mathbb{C}P^n$ called the <u>tautological line bundle</u> (or "universal subbundle" in Bott and Tu). Recall that $\mathbb{C}P^n$ can be thought of as the space of <u>complex</u> lines in $\mathbb{C}^{n+1}$. Consider the product bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$. An element of this space is a pair (ℓ, z) where ℓ is a line in $\mathbb{C}^{n+1}$ and z is a point in $\mathbb{C}^{n+1}$. The tautological bundle is defined as the set

$$L^n = \{(\ell, z) : z \in \ell\}.$$

   Compute the Euler class of L^1 by writing down transition functions for L^1.

4. (Exercise 6.46, abbreviated)

   (a) Let $i\colon S^n \to S^n$ be the antipodal map. An invariant differential form is a form ω so that $i^*\omega = \omega$. The space of such forms is closed under addition, wedge product, and exterior differentiation, so there is an <u>invariant cohomology</u> $H^*(S^n)^I$. Show that $H^*(\mathbb{R}P^n) \cong H^*(S^n)^I$.

   (b) It turns out that the natural map $H^*(S^n)^I \to H^*(S^n)$ is injective, but you don't need to prove that.

   (c) Show that i is orientation-preserving if and only if n is odd. Use this to conclude that, if $[\sigma]$ generates $H^n(S^n)$, then $[\sigma]$ is invariant if and only if n is odd.

   (d) Compute the de Rham cohomology of $\mathbb{R}P^n$:

$$H^q(\mathbb{R}P^n) \cong \begin{cases} \mathbb{R} & q = 0 \\ \mathbb{R} & q = n \text{ and } n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

5. A <u>division algebra</u> D is a non-necessarily-associative algebra over a field with the property that, for $a, b, x \in D$ and $b \neq 0$, if

$$a = bx$$

   then there is an element y so that

$$a = yb.$$

Suppose that we have some multiplication rule on $\mathbb{R}^n$ which makes it into a division algebra. Write e_i for the point $(0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0) \in \mathbb{R}^n$.

(a) Let $p \in S^{n-1} \subset \mathbb{R}^n$. Show that there is a unique $a \in \mathbb{R}^n$ so that $p = ae_1$.

(b) Let $b \in \mathbb{R}^n$ be non-zero. Show that $be_1, \ldots, be_n$ are linearly independent in $\mathbb{R}^n$.

(c) Let $p = ae_1$ as in the first bit. You can project $ae_2, \ldots, ae_n$ to $T_p(S^{n-1})$. Describe this projection and verify that the projections of $ae_2, \ldots, ae_n$ are still linearly independent.

(d) You may assume that multiplication by a fixed a is a continuous map. (But you are welcome to prove it.) Show that, if $\mathbb{R}^n$ can be given the structure of a division algebra, then TS^{n-1} is trivial. Conclude that $\mathbb{R}^3$ cannot have the structure of a division algebra.